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Postbuckling of Laminated Cylindrical Shells in Different Formulations

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The sensitivity of laminated cylindrical shells to imperfection is investigated throughout their entire nonlinear behavior. The study has two objectives: 1) comparison of the simplest formulation, using the Airy stress function (called the "WF formulation"), with the more accurate one, using the three displacement components (called the "UVW formulation"), and 2) examining the correlation between the sensitivity to imperfection according to this nonlinear analysis and that of Koiter's process for the initial postbuckling behavior. For laminated cylindrical shells—in contrast to isotropic shells—significant differences are observed between the two formulations (WF and UVW) and Koiter's theory does not always represent the actual sensitivity behavior. A general symbolic code (using MAPLE) was programmed to create the differential operators. Then the code used the Galerkin procedure, the Newton–Raphson procedure, and a finite difference scheme for automatic development of an efficient FORTRAN code, which was used for the parametric study.

Nomenclature

$[\boldsymbol{A}]$, $[\boldsymbol{B}]$, $[\boldsymbol{D}]$	=	membrane, coupling, and flexural rigidities
e_{av}	=	average end shortening
F	=	Airy stress function
$L(\,)$, $L(\,,\,)$, $L(\,,\,,\,)$	=	linear, quadratic, and cubic differential operators
$\{\boldsymbol{M}\}$, $\{\boldsymbol{N}\}$	=	bending moments, and membrane forces
N_u, N_v, N_w	=	retained terms of truncated Fourier series
$\bar{N}_{xx}$, $\bar{N}_{\theta\theta}$	=	external applied forces at the boundaries
n	=	characteristic circumferential wave number
Q_{ij}	=	laminate transformed reduced stiffness
q_w	=	external distributed loading in the normal direction
R	=	radius of the cylinder
u, v, w	=	axial, circumferential, and normal displacements
$\bar{w}$	=	initial geometric imperfection
x, θ, z	=	axial, circumferential, and outward normal coordinates of the mid surface
$(\,)_{,x}$, $(\,)_{,\theta}$	=	derivatives with respect to the axial and circumferential coordinate
Δs	=	arc length parameter
$\{\bar{\varepsilon}\}$, $\{\boldsymbol{\kappa}\}$	=	strain, and change of curvature of the reference surface
λ	=	load-level parameter

Introduction

LAMINATED cylindrical shells are already commonly used in structural engineering, and their buckling and postbuckling behavior is of vital importance in the design of such structures. The validity of linear buckling analysis in this context has been questioned because of the discrepancy observed between theoretical prediction and test results. The cause of this discrepancy is the fact that the nonlinear behavior of shell-like structures is generally characterized by a limit point rather than by a bifurcation point. For such structures, the load-carrying capacity depends on the level of imperfection (hence the concept of "imperfection sensitivity"). The motivation is, therefore, to reduce the sensitivity rather than preventing the imperfection. For that purpose, insight into the postbuckling state is called for.

There are two main solutions for reducing the sensitivity to imperfection: 1) built-up stiffened cylindrical shells with rings and axial stiffeners (see for example Hutchinson and Amazigo,[1] Hutchinson and Frauenthal,[2] Arbocz and Sechler,[3] Sheinman and Simitses,[4] and Simitses and Sheinman[5]); and 2) composite laminated layups (see for example Sheinman et al.,[6] Simitses et al.,[7] and Sheinman and Goldfeld[8]). In both, and especially in the second, the sensitivity should be investigated further theoretically and experimentally.

Two formulations, namely UVW and WF, are basically used in the cylindrical analysis. In UVW the unknown functions are the displacements in the axial (u), circumferential (v), and normal (w) directions, whereas WF involves only two functions, the normal displacement (w) and the Airy stress function (F). The advantage is offset by the fact that WF is applicable only in a Donnell-type shell theory. For laminated cylindrical shells, a significant difference was found between the two formulations for classical buckling.[9] Hence the motivation for extending it to postbuckling behavior.

From the analytical point of view, two approaches are used for investigating the sensitivity: 1) Parametric study of the shell in terms of its initial geometric imperfection, originally suggested by Koiter[10] and extensively studied by others (see the review paper of Simitses[11]). Recently, a comprehensive study, using that approach, of laminated cylindrical shells was published by Sheinman and Goldfeld.[8] 2) Tracing of the entire nonlinear equilibrium path with emphasis on the level and direction of change of the stiffness during loading. Using this approach, the sensitivity is measured by the load level at the limit point for a given imperfection amplitude and shape. Several research works have been done along these lines (see the review paper of Simitses[11] and Firer and Sheinman[12]), but they are confined to the WF formulation only. In addition, no correlation with approach 1 was examined. This approach entails a heavy computational effort, but due to the new generation of compilers and computers, it is applicable.

The present paper has two primary objectives for composite laminated cylindrical shells. First is comparison of the two formulations (UVW and WF) for the entire nonlinear behavior. According to the published paper of Shienman and Goldfeld,[9] it seems that for certain stacking combinations and orientations, the WF formulation is not representative of the entire nonlinear behavior. Second is examination of the correlation of the imperfection sensitivity between the initial postbuckling theory and the entire nonlinear behavior.

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Because of the involvement of the WF formulation, both objectives are investigated, based on a Donnell-type shell theory.

The nonlinear differential operators of the equilibrium equations are derived by applying the variational principle. A special symbolic algorithm, using a MAPLE compiler, was developed to derive the explicit form of the equilibrium equations. The solution procedure is based on expansion of the unknown functions (either u, v, and w, or w and F) in Fourier series in the circumferential direction and finite differences in the axial direction. Then the Galerkin procedure is applied to minimize the error due to truncation of the Fourier series. Finally, the arc-length algorithm is used to obtain the full nonlinear behavior. The whole programming procedure is written in a symbolic compiler which its output is the FORTRAN code, which is then used for the parametric study of the shell aspect ratio as well as of the lamination parameters.

Mathematical Formulation

Kinematics

The analytical model is based on the Kirchhoff–Love hypothesis, for which the strains at any material point (x, θ, z) (where x is a coordinate along the axial direction, θ the circumferential angle, and z the outward normal) read

$$\{\varepsilon(x,\theta,z)\} = \{\bar{\varepsilon}(x,\theta)\} + z\{\kappa(x,\theta)\} \tag{1}$$

Let $u(x,\theta)$, $v(x,\theta)$, and $w(x,\theta)$ be the components of the displacements surface of the shell in the x, θ, and z directions, respectively. Further, let $\bar{w}(x,\theta)$ be the initial geometric imperfection of the shell. Under the Donnell approach, the strain-displacement relations read

$$\{\bar{\varepsilon}(x,\theta)\} = \begin{Bmatrix} \bar{\varepsilon}_{xx} \\ \bar{\varepsilon}_{\theta\theta} \\ \bar{\gamma}_{x\theta} \end{Bmatrix} = \begin{Bmatrix} u_{,x} + \frac{1}{2}w_{,x}^2 + \bar{w}_{,x}w_{,x} \\ \frac{v_{,\theta}}{R} + \frac{w}{R} + \frac{w_{,\theta}^2}{2R^2} + \frac{\bar{w}_{,\theta}w_{,\theta}}{R^2} \\ \frac{u_{,\theta}}{R} + v_{,x} + \frac{w_{,x}w_{,\theta}}{R} + \frac{\bar{w}_{,x}w_{,\theta}}{R} + \frac{\bar{w}_{,\theta}w_{,x}}{R} \end{Bmatrix}$$

$$\{\boldsymbol{\kappa}(x,\theta)\} = \begin{Bmatrix} \kappa_{xx} \\ \kappa_{\theta\theta} \\ 2\kappa_{x\theta} \end{Bmatrix} = \begin{Bmatrix} -w_{,xx} \\ -\frac{w_{,\theta\theta}}{R^2} \\ -\frac{2w_{,x\theta}}{R} \end{Bmatrix} \tag{2}$$

$()_{,x}$ and $()_{,\theta}$ denote the derivatives with respect to the axial and circumferential coordinate, respectively; R is the radius of the cylinder.

Constitutive Relations

Under the classical laminate theory, the constitutive relations read

$$\begin{Bmatrix} \{\boldsymbol{N}\} \\ \{\boldsymbol{M}\} \end{Bmatrix} = \begin{bmatrix} [\boldsymbol{A}] & [\boldsymbol{B}] \\ [\boldsymbol{B}] & [\boldsymbol{D}] \end{bmatrix} \begin{Bmatrix} \{\bar{\varepsilon}\} \\ \{\boldsymbol{\kappa}\} \end{Bmatrix} \tag{3}$$

$\{\boldsymbol{N}\} = \{N_{xx}, N_{\theta\theta}, N_{x\theta}\}^T$ and $\{\boldsymbol{M}\} = \{M_{xx}, M_{\theta\theta}, M_{x\theta}\}^T$ being the resultant membrane forces and bending moments. The coefficients of the elastic matrix are given by

$$(A_{ij}, B_{ij}, D_{ij}) = \int_z Q_{ij}(1, z, z^2)\,\mathrm{d}z \tag{4}$$

A_{ij}, B_{ij}, and D_{ij} being, respectively, the membrane, coupling, and flexural rigidities, and Q_{ij} the laminate transformed reduced stiffness.

For the WF formulation, the strains $\{\bar{\varepsilon}\}$ and the bending moments $\{\boldsymbol{M}\}$ are expressed in terms of the membrane forces $\{\boldsymbol{N}\}$ and changes of curvature $\{\boldsymbol{\kappa}\}$ as

$$\begin{Bmatrix} \{\bar{\varepsilon}\} \\ \{\boldsymbol{M}\} \end{Bmatrix} = \begin{bmatrix} [\boldsymbol{a}] & -[\boldsymbol{b}] \\ [\boldsymbol{b}]^T & [\boldsymbol{d}] \end{bmatrix} \begin{Bmatrix} \{\boldsymbol{N}\} \\ \{\boldsymbol{\kappa}\} \end{Bmatrix} \tag{5}$$

where $[\boldsymbol{a}] = [\boldsymbol{A}]^{-1}$, $[\boldsymbol{b}] = [\boldsymbol{A}]^{-1}[\boldsymbol{B}]$, and $[\boldsymbol{d}] = [\boldsymbol{D}] - [\boldsymbol{B}][\boldsymbol{A}]^{-1}[\boldsymbol{B}]$.

Equilibrium Equations

The equilibrium equations and the appropriate boundary conditions are derived from the variational principle

$$\delta\pi = \int_A [N_{xx}\delta\bar{\varepsilon}_{xx} + N_{\theta\theta}\delta\bar{\varepsilon}_{\theta\theta} + N_{x\theta}\delta\bar{\gamma}_{x\theta} + M_{xx}\delta\kappa_{xx} + M_{\theta\theta}\delta\kappa_{\theta\theta} + 2M_{x\theta}\delta\kappa_{x\theta} - q_w\delta w]\,\mathrm{d}A \tag{6}$$

where q_w is the external distributed loading in the radial direction. With Eqs. (2) substituted into Eq. (6), application of Gauss's theorem yields the following equilibrium equations:

$$N_{xx,x} + \frac{N_{x\theta,\theta}}{R} = 0$$

$$\frac{N_{\theta\theta,\theta}}{R} + N_{x\theta,x} = 0$$

$$M_{xx,xx} + \frac{2M_{x\theta,x\theta}}{R} + \frac{M_{\theta\theta,\theta\theta}}{R^2} - \frac{N_{\theta\theta}}{R} + \left[N_{xx}(w_{,x} + \bar{w}_{,x}) + \frac{N_{x\theta}(w_{,\theta} + \bar{w}_{,\theta})}{R}\right]_{,x} + \left[\frac{N_{x\theta}(w_{,x} + \bar{w}_{,x})}{R} + \frac{N_{\theta\theta}(w_{,\theta} + \bar{w}_{,\theta})}{R^2}\right]_{,\theta} + q_w = 0 \tag{7}$$

with the following boundary conditions:

$$\begin{array}{lll} N_{xx} & \text{or} & u \\ N_{x\theta} & \text{or} & v \\ M_{xx,x} + \frac{2M_{x\theta,\theta}}{R} + N_{xx}(w_{,x} + \bar{w}_{,x}) + \frac{N_{x\theta}(w_{,\theta} + \bar{w}_{,\theta})}{R} & \text{or} & w \\ M_{xx} & \text{or} & w_{,x} \end{array} \tag{8}$$

Using the kinematic and constitutive relations [Eqs. (2) and (3)], the equilibrium equations [Eqs. (7)] can be written in terms of the displacement components as

$$\phi_p(u, v, w) = 0, \qquad p = 1, 2, 3 \tag{9}$$

where ϕ_p consists of differential operators:

$$L_p^1(u) + L_p^2(v) + L_p^3(w) + L_p^4(\bar{w}, u) + L_p^5(\bar{w}, v) + L_p^6(\bar{w}, w) + L_p^7(u, w) + L_p^8(v, w) + L_p^9(w, w) + L_p^{10}(\bar{w}, \bar{w}, w) + L_p^{11}(\bar{w}, w, w) + L_p^{12}(w, w, w) + q_p = 0, \qquad p = 1, 2, 3 \tag{10}$$

where $L_p^e(Q)$, $L_p^e(Q, S)$, and $L_p^e(Q, S, T)$ are the linear, quadratic, and cubic differential operators,[9] given by

$$L_p^e(Q) = \sum_{i=0}^{4}\sum_{j=0}^{4-i} \Re_{ij}^{p,e} \frac{\partial^{(i+j)}Q}{\partial x^{(i)}\partial\theta^{(j)}}$$

$$L_p^e(Q, S) = \sum_{i=0}^{3}\sum_{j=0}^{3-i}\sum_{k=0}^{3}\sum_{\ell=0}^{\ell-k} \Re_{ijk\ell}^{p,e} \frac{\partial^{(i+j)}Q}{\partial x^{(i)}\partial\theta^{(j)}} \frac{\partial^{(k+\ell)}S}{\partial x^{(k)}\partial\theta^{(\ell)}}$$

$$L_p^e(Q, S, T) = \sum_{i=0}^{2}\sum_{j=0}^{2-i}\sum_{k=0}^{2}\sum_{\ell=0}^{2-k}\sum_{m=0}^{2}\sum_{n=0}^{2-m} \Re_{ijk\ell mn}^{p,e} \frac{\partial^{(i+j)}Q}{\partial x^{(i)}\partial\theta^{(j)}} \times \frac{\partial^{(k+\ell)}S}{\partial x^{(k)}\partial\theta^{(\ell)}} \frac{\partial^{(m+n)}T}{\partial x^{(m)}\partial\theta^{(n)}} \tag{11}$$

Table 1 Number of differential operators in equilibrium and boundary conditions equations (relevant equation in parentheses)

Equilibrium equation			Boundary conditions			
			UVW			
30 (7a)	30 (7b)	148 (7c)	15 (8a)	15 (8b)	62 (8c)	15 (8d)
			WF			
16 (13)	22 (14)					

where $\Re_{ij}^{p,e}$, $\Re_{ijk\ell}^{p,e}$, and $\Re_{ijk\ell mn}^{p,e}$ are coefficients of the elastic parameters, functions of A_{ij}, B_{ij}, D_{ij}, and the radius R. The explicit form of the differential operators is given in the Appendix. The boundary conditions (Eqs. (8)) are written in a similar way. This form is especially suitable for symbolic programming.

The WF formulation is obtained by introducing the Airy stress function $F(x,\theta)$,

$$\{N_{xx}, N_{\theta\theta}, N_{x\theta}\} = \left\{F_{,\theta\theta}/R^2 + \bar{N}_{xx},\ F_{,xx},\ -F_{,x\theta}/R + \bar{N}_{x\theta}\right\} \tag{12}$$

where $\bar{N}_{xx}$ and $\bar{N}_{x\theta}$ are the external applied forces at the boundaries. Then, the relevant equations are the compatibility equation

$$L_q(w) + L_g(F) + \tfrac{1}{2}L_{\mathrm{NL}}(w + 2\bar{w}, w) - (w_{,xx})/R = 0 \tag{13}$$

and the equilibrium equation

$$L_h(w) + L_q(F) + L_{\mathrm{NL}}(w + \bar{w}, F) - F_{,xx}/R + \bar{N}_{xx}(w_{,xx} + \bar{w}_{,xx}) + 2(\bar{N}_{x\theta}/R)(w_{,x\theta} + \bar{w}_{,x\theta}) + q_w = 0 \tag{14}$$

The differential operators in Eqs. (13) and (14) are defined implicitly in Ref. 9 and explicitly in Ref. 6.

The advantage of WF over its UVW counterpart lies mainly in reducing the number of unknown functions (from 3 to 2) and that of differential operators (Table 1). On the other hand, it is restricted to Donnell-type equations, and the conditions on u and v are expressed in terms of w and F, which can lead to a discrepancy in the limit point load, especially for a composite laminated cylindrical shell.

Solution Procedure

The set of partial differential equations (7) and (8) is reduced to one of ordinary differential equations by separation of variables and expansion into truncated Fourier series as

$$\{u(x,\theta), v(x,\theta), w(x,\theta)\} = \sum_{m=0}^{2N}\{u_m(x), v_m(x), w_m(x)\}g_m(\theta) \tag{15}$$

where $N = N_u$ or N_v or N_w is the number of terms in the relevant series. The initial geometric imperfection and the external load q_w are

$$\bar{w}(x,\theta) = \sum_{m=0}^{2N_w}\bar{w}_m(x)g_m(\theta) \tag{16}$$

$$q_w(x,\theta) = \sum_{m=0}^{2N_q}q_m(x)g_m(\theta) \tag{17}$$

The functions $g_m(\theta)$ are

$$g_m(\theta) = \begin{cases}\cos(nm\theta), & m = 0, 1, \ldots, N\\ \sin[n(m-N)\theta], & m = N+1, \ldots, 2N\end{cases} \tag{18}$$

where n denotes the characteristic circumferential wave number. Recourse to the latter makes it possible, in some cases, to reduce substantially the number of terms in the Fourier series.[13] For general cases, in which all terms are significant, it is necessary to let $n = 1$ and N be sufficiently large for an accurate representation of u, v, and w.

The equilibrium set consists of fourth-order differential equations. These were converted into six equations of second order in three new unknown functions:

$$\{\xi^u, \xi^v, \xi^w\} = \{u_{,xx}, v_{,xx}, w_{,xx}\} \tag{19}$$

Minimization of the errors due to the truncated Fourier series by the Galerkin procedure with cos( ) and sin( ) as weighting functions yields the following nonlinear ordinary differential equations:

$$\Phi_r^q(z; x) = \oint \phi_r(u, v, w)g_q(\theta)\,\mathrm{d}\theta$$
$$q = 0, 1, \ldots, 2N, \qquad r = 1, 2, \ldots, 6 \tag{20}$$

where Φ_r^q comprises $4(N_u + N_v + N_w) + 6$ nonlinear ordinary differential equations, and z is an unknown vector function, namely,

$$z = \begin{Bmatrix} u_0, \ldots, u_{2N_u}, v_0, \ldots, v_{2N_v}, w_0, \ldots, w_{2N_w},\\ \xi_0^u, \ldots, \xi_{2N_u}^u, \xi_0^v, \ldots, \xi_{2N_v}^v, \xi_0^w, \ldots, \xi_{2N_w}^w \end{Bmatrix}^T \tag{21}$$

Equation (20) is approximated by finite differences in the form

$$\boldsymbol{G}(\boldsymbol{u}, \lambda) = \boldsymbol{0} \tag{22}$$

where $\boldsymbol{G}$ consists of nonlinear algebraic operators, $\boldsymbol{u}$ is the value of the unknown functions at each point of the finite-difference scheme, and λ is the load-level parameter. This equilibrium path is obtained with the aid of the constant arc-length algorithm,[14] with the additional constraint

$$N(\boldsymbol{u}, \lambda) \equiv \left\{\frac{\mathrm{d}\boldsymbol{u}_0}{\mathrm{d}s}\right\}^T\{\boldsymbol{u} - \boldsymbol{u}_0\} + \frac{\mathrm{d}\lambda_0}{\mathrm{d}s}(\lambda - \lambda_0) - \Delta s = 0 \tag{23}$$

where $\{\boldsymbol{u}_0, \lambda_0\}$ is a point on the equilibrium path, $\{\mathrm{d}\boldsymbol{u}_0/\mathrm{d}s, \mathrm{d}\lambda_0/\mathrm{d}s\}$ is the unit tangent to the path at this point, and Δs is the arc length.

One of the difficulties involved in analyzing Eqs. (9) is the large number of differential operators (summarized in Table 1), for each of which the Galerkin procedure, the Newton–Raphson procedure, and discretization by finite differences have to be implemented. This situation motivates recourse to a general symbolic algorithm, using a MAPLE compiler. The equilibrium equations in terms of the generalized forces, and the constitutive relations, are introduced into the symbolic program and the following stages are implemented: 1) Substitution of the constitutive relations into the equilibrium equation, yields the equations in terms of the unknown displacements. The program then identifies the different operators by identifying the displacement functions and calculating their coefficients, which are the first output of the program. 2) Each identified operator is multiplied by a weighting function and the Galerkin procedure is implemented. The symbolic result of the integral, Galerkin, is defined and automatically outputted into a FORTRAN code. 3) The Newton–Raphson procedure is implemented.

Results and Discussion

For the procedure outlined above, a special purpose computer code, NALICS (nonlinear analysis of laminated imperfect cylindrical shells), was developed, covering the entire nonlinear behavior of any shell of the above type under arbitrary axial, torsional, and hydrostatic-pressure loading. The code was checked out by comparison of the nonlinear behavior with ANSYS (a commercial finite elements code) for isotropic cylindrical shells, and with Sheinman et al.[6] for laminated cylindrical shells (using the WF formulation). In both the results were in very good agreement.

The present parametric study has two primary objectives: 1) Comparison of the UVW and WF formulations over the entire nonlinear behavior and 2) examination of the correlation of the imperfection sensitivity over the entire nonlinear behavior with that of the initial postbuckling behavior (Koiter's theory).

The example, reproduced from Sheinman and Goldfeld,[9] concerns an angle-ply ($\pm\theta$) graphite/epoxy cylindrical shell under axisymmetric axial compression. The data are as follows:

two-ply laminate with elastic modulus: $E_{11} = 1.404 \times 10^{11}$ N/m^2, $E_{22} = 0.973 \times 10^{10}$ N/m^2 ($E_{11}/E_{22} = 14.4$), shear modulus $G_{12} = 0.411 \times 10^{10}$ N/m^2 ($E_{11}/G_{12} = 34.1$), Poission's ratio $\nu_{12} = 0.26$, thickness $h = 0.0127$ m, radius $R = 1.27$ m, ($R/h = 100$), length $L = 2.54$ m, ($L/R = 2$). Boundary conditions: out-of-plane, $w = w_{,x} = 0$ at both ends; in-plane (for the UVW formulation), $u = v = 0$ at one end, $N_{xx} = -\bar{N}_{xx}$ and $N_{x\theta} = 0$ (unless $v = 0$ explicitly specified) at the other. For the WF formulation, the boundary conditions are clamped–clamped: CC$_4$ ($w = w_{,x} = u = v = 0$) at one end and CC$_1$ ($w = w_{,x} = N_{x\theta} = 0$, $N_{xx} = -\bar{N}_{xx}$) at the other. Initial symmetric imperfection [$\bar{w}(x,\theta) = \xi h \sin(\pi x/L)\cos(n\theta)$] is assumed.

Comparison of UVW and WF Formulations

The axial compression $\bar{N}_{xx}$ vs the average end shortening,

$$e_{\mathrm{av}} = \frac{-\int_A u_{,x}\,\mathrm{d}A}{(2\pi RL)}$$

is plotted in Fig. 1 for imperfection amplitude $\xi = 0.1$. The results are generated for several values of the characteristic circumferential wave number, n. The load level at the limit point, $\bar{N}_{xx,\mathrm{cr}}$, is definitely n-dependent, and it is observed that the lowest limit point is the smallest of all values corresponding to different wave numbers n.

The postbuckling branch shows several changes from $n = 7$ to $n = 6$ to $n = 5$ to $n = 4$. In reality, the system reaches the limit point and then snaps through (violent buckling) toward a far stable equilibrium position with a sharp change of the circumferential wave number.

Comparison of UVW and WF for the orthotropic case, (± 0 deg) is given in Fig. 2. It is seen that the two formulations yield the same response with only a small difference far along the postbuckling path. For laminate material ($\pm\theta$ deg) a discrepancy is seen in Fig. 3 (for ± 30 deg) with differences in the postbuckling behavior, and in Fig. 4 (for ± 60 deg) with differences even along the primary path, including the nonlinear prebuckling stage.

The limit point in both formulations versus the angle-ply ($\pm\theta$) is plotted in Fig. 5. It is seen that the difference between the formulations is θ-dependent. The discrepancy is significant in the interval 30 deg $< \theta <$ 70 deg.

Correlation Between the Entire Nonlinear and the Initial (Koiter's Theory) Postbuckling Behavior

For examining the correlation the initial slope of the secondary path associated with the full nonlinear behavior is compared with the b-Koiter parameter, given by Sheinman and Goldfeld.[8]

In view of the pronounced effect of the in-plane boundary condition as shown for the initial postbuckling approach by Sheinman and Goldfeld,[8] the correlation was checked out for both $N_{x\theta} = 0$ and $v = 0$. The normalized results, given in Fig. 6 (left ordinate relates to b-Koiter and right ordinate to initial slope of the secondary path), show a very good correlation in terms of the sensitivity behavior. Both approaches show sensitivity behavior over the entire angle-ply range, for $N_{x\theta} = 0$, and insensitivity over most of the range except the 0 deg $\leq \theta \leq$ 3 deg interval, for $v = 0$.

As regards the insensitivity region, it is apparently confined to the vicinity of the bifurcation point. For example, for $v = 0$ with $\theta = \pm 15$ deg the behavior is insensitive and away from the original bifurcation it becomes sensitive as shown in Fig. 7. Note that this pattern can be revealed only through the nonlinear approach, but not through Koiter's theory. In other words, a shell can be rated as insensitive according to Koiter's theory, whereas in reality it is sensitive.

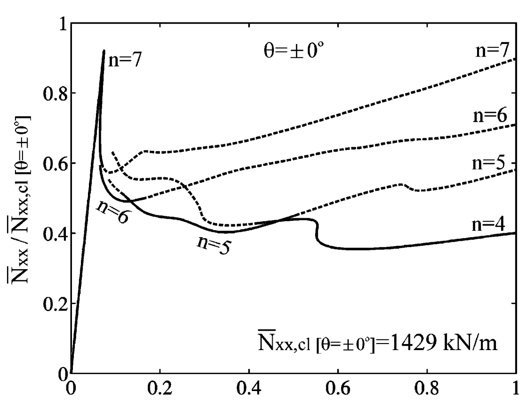

Fig. 1 Axial load vs average end shortening for θ = ±0 deg, UVW formulation.

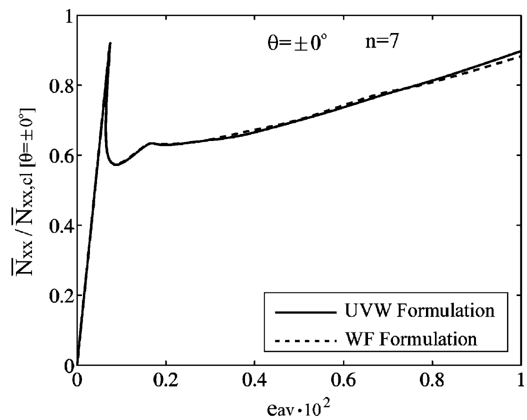

Fig. 2 Comparison of UVW and WF with regard to nonlinear behavior, orthotropic configuration (θ = ±0 deg).

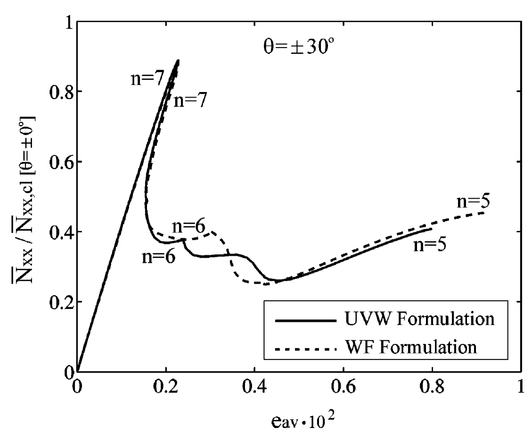

Fig. 3 Comparison of UVW and WF with regard to nonlinear behavior, laminate configuration (θ = ±30 deg).

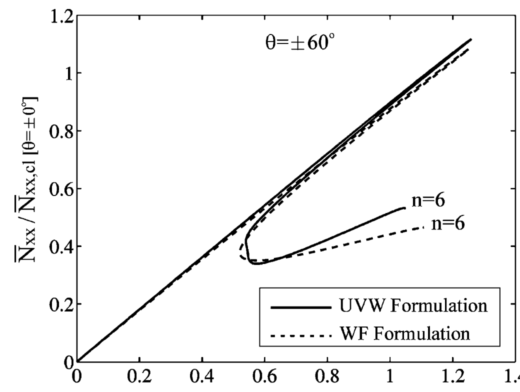

Fig. 4 Comparison of UVW and WF with regard to nonlinear behavior, laminate configuration (θ = ±60 deg).

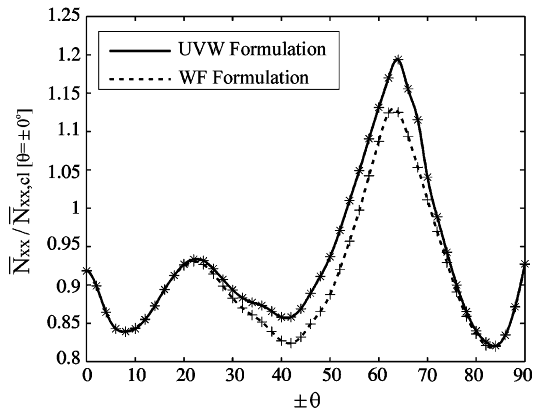

Fig. 5 Comparison of UVW and WF with regard to limit point.

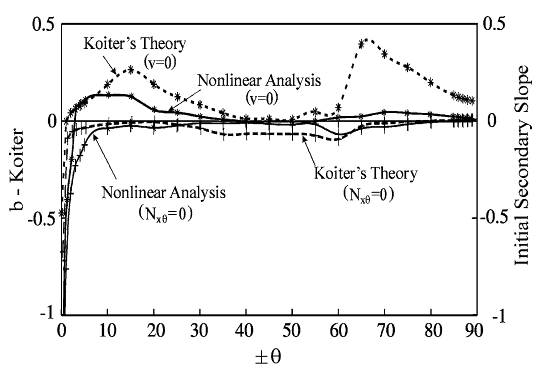

Fig. 6 Correlation between nonlinear analysis and Koiter's theory.

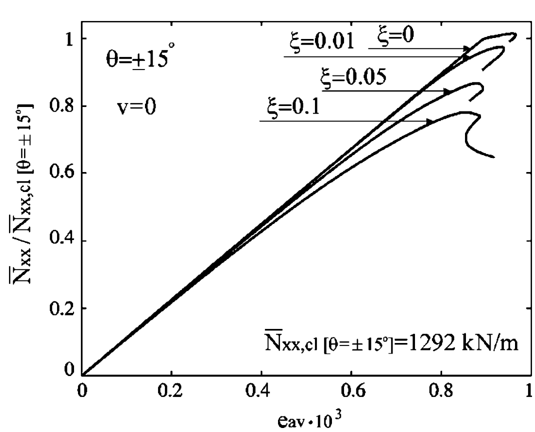

Fig. 7 Sensitivity path with respect to imperfection amplitude (ξ) for $\theta = \pm 15$ deg.

Conclusions

An automatic symbolic code creating a numerical procedure is presented for laminated cylindrical shells using the UVW and WF formulations. The code is used for comparison of the formulations and examination of the correlation with Koiter's theory. From the results the following findings are concluded:

1) For laminated cylindrical shells—in contrast to the isotropic case—the WF formulation is not always representative; the real limit point and the postbuckling behavior have to be considered by the UVW.

2) A correlation of the sensitivity to imperfection near the bifurcation point is observed between Koiter's process and the more accurate one (full nonlinear analysis).

3) Koiter's theory is confined to the vicinity of the bifurcation point. The real behavior can be determined a little away from the bifurcation point and is obtainable only by the full nonlinear process.

Appendix: Differential Operators for UVW

The explicit form of the differential operators reads as follows:

$$L_1^1(u) = \frac{A_{33}}{R^2}u_{,\theta\theta} + 2\frac{A_{13}}{R}u_{,x\theta} + A_{11}u_{,xx} \tag{A1}$$

$$L_1^2(v) = \frac{A_{23}}{R^2}v_{,\theta\theta} + \frac{A_{12}+A_{33}}{R}v_{,x\theta} + A_{13}v_{,xx} \tag{A2}$$

$$L_1^3(w) = \frac{A_{23}}{R^2}w_{,\theta} - \frac{B_{23}}{R^3}w_{,\theta\theta\theta} + \frac{A_{12}}{R}w_{,x} - \frac{B_{12}+2B_{33}}{R^2}w_{,x\theta\theta} - \frac{3B_{13}}{R}w_{,xx\theta} - B_{11}w_{,xxx} \tag{A3}$$

$$L_1^4(\bar{w}, u) = 0 \tag{A4}$$

$$L_1^5(\bar{w}, v) = 0 \tag{A5}$$

$$L_1^6(\bar{w}, w) = \frac{A_{23}}{R^3}\bar{w}_{,\theta}w_{,\theta\theta} + \frac{A_{12}+A_{33}}{R^2}\bar{w}_{,\theta}w_{,x\theta} + \frac{A_{13}}{R}\bar{w}_{,\theta}w_{,xx} + \frac{A_{23}}{R^3}\bar{w}_{,\theta\theta}w_{,\theta} + \frac{A_{33}}{R^2}\bar{w}_{,\theta\theta}w_{,x} + \frac{A_{33}}{R^2}\bar{w}_{,x}w_{,\theta\theta} + \frac{2A_{13}}{R}\bar{w}_{,x}w_{,x\theta} + A_{11}\bar{w}_{,x}w_{,xx} + \frac{A_{12}+A_{33}}{R^2}\bar{w}_{,x\theta}w_{,\theta} + \frac{2A_{13}}{R}\bar{w}_{,x\theta}w_{,x} + \frac{A_{13}}{R}\bar{w}_{,xx}w_{,\theta} + A_{11}\bar{w}_{,xx}w_{,x} \tag{A6}$$

$$L_1^7(u, w) = 0 \tag{A7}$$

$$L_1^8(v, w) = 0 \tag{A8}$$

$$L_1^9(w, w) = \frac{A_{23}}{R^3}w_{,\theta}w_{,\theta\theta} + \frac{A_{12}+A_{33}}{R^2}w_{,\theta}w_{,x\theta} + \frac{A_{13}}{R}w_{,\theta}w_{,xx} + \frac{A_{33}}{R^2}w_{,\theta\theta}w_{,x} + \frac{2A_{13}}{R}w_{,x}w_{,x\theta} + A_{11}w_{,x}w_{,xx} \tag{A9}$$

$$L_1^{10}(\bar{w}, \bar{w}, w) = 0 \tag{A10}$$

$$L_1^{11}(\bar{w}, w, w) = 0 \tag{A11}$$

$$L_1^{12}(w, w, w) = 0 \tag{A12}$$

$$L_2^1(u) = \frac{A_{23}}{R^2}u_{,\theta\theta} + \frac{A_{12}+A_{33}}{R}u_{,x\theta} + A_{13}u_{,xx} \tag{A13}$$

$$L_2^2(v) = \frac{A_{22}}{R^2}v_{,\theta\theta} + \frac{2A_{23}}{R}v_{,x\theta} + A_{33}v_{,xx} \tag{A14}$$

$$L_2^3(w) = \frac{A_{22}}{R^2}w_{,\theta} - \frac{B_{22}}{R^3}w_{,\theta\theta\theta} + \frac{A_{23}}{R}w_{,x} - \frac{3B_{23}}{R^2}w_{,x\theta\theta} - \frac{B_{12}+2B_{33}}{R}w_{,xx\theta} - B_{13}w_{,xxx} \tag{A15}$$

$$L_2^4(\bar{w}, u) = 0 \tag{A16}$$

$$L_2^5(\bar{w}, v) = 0 \tag{A17}$$

$$L_2^6(\bar{w}, w) = \frac{A_{22}}{R^3}\bar{w}_{,\theta}w_{,\theta\theta} + \frac{2A_{23}}{R^2}\bar{w}_{,\theta}w_{,x\theta} + \frac{A_{33}}{R}\bar{w}_{,\theta}w_{,xx} + \frac{A_{22}}{R^3}\bar{w}_{,\theta\theta}w_{,\theta} + \frac{A_{23}}{R^2}\bar{w}_{,\theta\theta}w_{,x} + \frac{A_{23}}{R^2}\bar{w}_{,x}w_{,\theta\theta} + \frac{A_{12}+A_{33}}{R}\bar{w}_{,x}w_{,x\theta} + A_{13}\bar{w}_{,x}w_{,xx} + \frac{2A_{23}}{R^2}\bar{w}_{,x\theta}w_{,\theta} + \frac{A_{12}+A_{33}}{R}\bar{w}_{,x\theta}w_{,x} + \frac{A_{33}}{R}\bar{w}_{,xx}w_{,\theta} + A_{13}\bar{w}_{,xx}w_{,x} \tag{A18}$$

$$L_2^7(u, w) = 0 \quad \text{(A19)}$$

$$L_2^8(v, w) = 0 \quad \text{(A20)}$$

$$L_2^9(w, w) = \frac{A_{22}}{R^3} w_{,\theta} w_{,\theta\theta} + \frac{2A_{23}}{R^2} w_{,\theta} w_{,x\theta} + \frac{A_{33}}{R} w_{,\theta} w_{,xx} + \frac{A_{23}}{R^2} w_{,\theta\theta} w_{,x} + \frac{A_{12} + A_{33}}{R} w_{,x} w_{,x\theta} + A_{13} w_{,x} w_{,xx} \quad \text{(A21)}$$

$$L_2^{10}(\bar{w}, \bar{w}, w) = 0 \quad \text{(A22)}$$

$$L_2^{11}(\bar{w}, w, w) = 0 \quad \text{(A23)}$$

$$L_2^{12}(w, w, w) = 0 \quad \text{(A24)}$$

$$L_3^1(u) = -\frac{A_{23}}{R^2} u_{,\theta} + \frac{B_{23}}{R^3} u_{,\theta\theta\theta} - \frac{A_{12}}{R} u_{,x} + \frac{B_{12} + 2B_{33}}{R^2} u_{,x\theta\theta} + \frac{3B_{13}}{R} u_{,xx\theta} + B_{11} u_{,xxx} \quad \text{(A25)}$$

$$L_3^2(v) = -\frac{A_{22}}{R^2} v_{,\theta} + \frac{B_{22}}{R^3} v_{,\theta\theta\theta} - \frac{A_{23}}{R} v_{,x} + \frac{3B_{23}}{R^2} v_{,x\theta\theta} + \frac{B_{12} + 2B_{33}}{R} v_{,xx\theta} + B_{13} v_{,xxx} \quad \text{(A26)}$$

$$L_3^3(w) = -\frac{A_{22}}{R^2} w + \frac{2B_{22}}{R^3} w_{,\theta\theta} - \frac{D_{22}}{R^4} w_{,\theta\theta\theta\theta} + \frac{4B_{23}}{R^2} w_{,x\theta} - \frac{4D_{23}}{R^3} w_{,x\theta\theta\theta} + \frac{2B_{12}}{R} w_{,xx} - 2\frac{D_{12} + 2D_{33}}{R^2} w_{,xx\theta\theta} - \frac{4D_{13}}{R} w_{,xxx\theta} - D_{11} w_{,xxxx} \quad \text{(A27)}$$

$$L_3^4(\bar{w}, u) = \frac{A_{23}}{R^3} \bar{w}_{,\theta} u_{,\theta\theta} + \frac{A_{12} + A_{33}}{R^2} \bar{w}_{,\theta} u_{,x\theta} + \frac{A_{13}}{R} \bar{w}_{,\theta} u_{,xx} + \frac{A_{23}}{R^3} \bar{w}_{,\theta\theta} u_{,\theta} + \frac{A_{12}}{R^2} \bar{w}_{,\theta\theta} u_{,x} + \frac{A_{33}}{R^2} \bar{w}_{,x} u_{,\theta\theta} + \frac{2A_{13}}{R} \bar{w}_{,x} u_{,x\theta} + A_{11} \bar{w}_{,x} u_{,xx} + \frac{2A_{33}}{R^2} \bar{w}_{,x\theta} u_{,\theta} + \frac{2A_{13}}{R} \bar{w}_{,x\theta} u_{,x} + \frac{A_{13}}{R} \bar{w}_{,xx} u_{,\theta} + A_{11} \bar{w}_{,xx} u_{,x} \quad \text{(A28)}$$

$$L_3^5(\bar{w}, v) = \frac{A_{22}}{R^3} \bar{w}_{,\theta} v_{,\theta\theta} + \frac{2A_{23}}{R^2} \bar{w}_{,\theta} v_{,x\theta} + \frac{A_{33}}{R} \bar{w}_{,\theta} v_{,xx} + \frac{A_{22}}{R^3} \bar{w}_{,\theta\theta} v_{,\theta} + \frac{A_{23}}{R^2} \bar{w}_{,\theta\theta} v_{,x} + \frac{A_{23}}{R^2} \bar{w}_{,x} v_{,\theta\theta} + \frac{A_{12} + A_{33}}{R} \bar{w}_{,x} v_{,x\theta} + A_{13} \bar{w}_{,x} v_{,xx} + \frac{2A_{23}}{R^2} \bar{w}_{,x\theta} v_{,\theta} + \frac{2A_{33}}{R} \bar{w}_{,x\theta} v_{,x} + \frac{A_{12}}{R} \bar{w}_{,xx} v_{,\theta} + A_{13} \bar{w}_{,xx} v_{,x} \quad \text{(A29)}$$

$$L_3^6(\bar{w}, w) = \frac{A_{22}}{R^3} \bar{w}_{,\theta\theta} w + \frac{B_{22}}{R^4} \bar{w}_{,\theta\theta} w_{,\theta\theta} + \frac{2B_{23}}{R^3} \bar{w}_{,\theta\theta} w_{,x\theta} + \frac{2B_{33} - B_{12}}{R^2} \bar{w}_{,\theta\theta} w_{,xx} + \frac{B_{22}}{R^4} \bar{w}_{,\theta\theta\theta} w_{,\theta} + \frac{B_{23}}{R^3} \bar{w}_{,\theta\theta\theta} w_{,x} + \frac{2A_{23}}{R^2} \bar{w}_{,x\theta} w + \frac{2B_{23}}{R^3} \bar{w}_{,x\theta} w_{,\theta\theta} + \frac{4B_{12}}{R^2} \bar{w}_{,x\theta} w_{,x\theta} + \frac{2B_{13}}{R} \bar{w}_{,x\theta} w_{,xx} + \frac{3B_{23}}{R^3} \bar{w}_{,x\theta\theta} w_{,\theta} + \frac{B_{12} + 2B_{33}}{R^2} \bar{w}_{,x\theta\theta} w_{,x} + \frac{A_{12}}{R} \bar{w}_{,xx} w + \frac{2B_{33} - B_{12}}{R^2} \bar{w}_{,xx} w_{,\theta\theta} + \frac{2B_{13}}{R} \bar{w}_{,xx} w_{,x\theta} + B_{11} \bar{w}_{,xx} w_{,xx} + \frac{B_{12} + 2B_{33}}{R^2} \bar{w}_{,xx\theta} w_{,\theta} + \frac{3B_{13}}{R} \bar{w}_{,xx\theta} w_{,x} + \frac{B_{13}}{R} \bar{w}_{,xxx} w_{,\theta} + B_{11} \bar{w}_{,xxx} w_{,x} \quad \text{(A30)}$$

$$L_3^7(u, w) = \frac{A_{23}}{R^3} u_{,\theta} w_{,\theta\theta} + \frac{2A_{33}}{R^2} u_{,\theta} w_{,x\theta} + \frac{A_{13}}{R} u_{,\theta} w_{,xx} + \frac{A_{23}}{R^3} u_{,\theta\theta} w_{,\theta} + \frac{A_{33}}{R^2} u_{,\theta\theta} w_{,x} + \frac{A_{12}}{R^2} u_{,x} w_{,\theta\theta} + \frac{2A_{13}}{R} u_{,x} w_{,x\theta} + A_{11} u_{,x} w_{,xx} + \frac{A_{12} + A_{33}}{R^2} u_{,x\theta} w_{,\theta} + \frac{2A_{13}}{R} u_{,x\theta} w_{,x} + \frac{A_{13}}{R} u_{,xx} w_{,\theta} + A_{11} u_{,xx} w_{,x} \quad \text{(A31)}$$

$$L_3^8(v, w) = \frac{A_{22}}{R^3} v_{,\theta} w_{,\theta\theta} + \frac{2A_{23}}{R^2} v_{,\theta} w_{,x\theta} + \frac{A_{12}}{R} v_{,\theta} w_{,xx} + \frac{A_{22}}{R^3} v_{,\theta\theta} w_{,\theta} + \frac{A_{23}}{R^2} v_{,\theta\theta} w_{,x} + \frac{A_{23}}{R^2} v_{,x} w_{,\theta\theta} + \frac{2A_{33}}{R} v_{,x} w_{,x\theta} + A_{13} v_{,x} w_{,xx} + \frac{2A_{23}}{R^2} v_{,x\theta} w_{,\theta} + \frac{A_{12} + A_{33}}{R} v_{,x\theta} w_{,x} + \frac{A_{33}}{R} v_{,xx} w_{,\theta} + A_{13} v_{,xx} w_{,x} \quad \text{(A32)}$$

$$L_3^9(w, w) = \frac{A_{22}}{R^3} w w_{,\theta\theta} + \frac{2A_{23}}{R^2} w w_{,x\theta} + \frac{A_{12}}{R} w w_{,x} + \frac{A_{22}}{2R^3} w_{,\theta}^2 + \frac{A_{23}}{R^2} w_{,\theta} w_{,x} + 2\frac{B_{33} - B_{12}}{R^2} w_{,\theta\theta} w_{,xx} + \frac{A_{12}}{2R} w_{,x}^2 + 2\frac{B_{12} - B_{33}}{R^2} w_{,x\theta}^2 \quad \text{(A33)}$$

$$L_3^{10}(\bar{w}, \bar{w}, w) = \frac{A_{22}}{R^4} \bar{w}_{,\theta}^2 w_{,\theta\theta} + \frac{2A_{23}}{R^3} \bar{w}_{,\theta}^2 w_{,x\theta} + \frac{A_{33}}{R^2} \bar{w}_{,\theta}^2 w_{,xx} + \frac{2A_{22}}{R^4} \bar{w}_{,\theta} \bar{w}_{,\theta\theta} w_{,\theta} + \frac{2A_{23}}{R^3} \bar{w}_{,\theta} \bar{w}_{,\theta\theta} w_{,x} + \frac{2A_{23}}{R^3} \bar{w}_{,\theta} \bar{w}_{,x} w_{,\theta\theta} + 2\frac{A_{12} + A_{33}}{R^2} \bar{w}_{,\theta} \bar{w}_{,x} w_{,x\theta} + \frac{2A_{13}}{R} \bar{w}_{,\theta} \bar{w}_{,x} w_{,xx} + \frac{4A_{23}}{R^3} \bar{w}_{,\theta} \bar{w}_{,x\theta} w_{,\theta} + \frac{A_{12} + 3A_{33}}{R^2} \bar{w}_{,\theta} \bar{w}_{,x\theta} w_{,x} + \frac{A_{12} + A_{33}}{R^2} \bar{w}_{,\theta} \bar{w}_{,xx} w_{,\theta} + \frac{2A_{13}}{R} \bar{w}_{,\theta} \bar{w}_{,xx} w_{,x} + \frac{2A_{23}}{R^3} \bar{w}_{,\theta\theta} \bar{w}_{,x} w_{,\theta} + \frac{A_{12} + A_{33}}{R^2} \bar{w}_{,\theta\theta} \bar{w}_{,x} w_{,x} + \frac{A_{33}}{R^2} \bar{w}_{,x}^2 w_{,\theta\theta} + \frac{2A_{13}}{R} \bar{w}_{,x}^2 w_{,x\theta} + A_{11} \bar{w}_{,x}^2 w_{,xx} + \frac{A_{12} + 3A_{33}}{R^2} \bar{w}_{,x} \bar{w}_{,x\theta} w_{,\theta} + \frac{4A_{13}}{R} \bar{w}_{,x} \bar{w}_{,x\theta} w_{,x} + \frac{2A_{13}}{R} \bar{w}_{,x} \bar{w}_{,xx} w_{,\theta} + 2A_{11} \bar{w}_{,x} \bar{w}_{,xx} w_{,x} \quad \text{(A34)}$$

$$L_3^{11}(\bar{w}, w, w) = \frac{3A_{22}}{R^4} \bar{w}_{,\theta} w_{,\theta} w_{,\theta\theta} + \frac{6A_{23}}{R^3} \bar{w}_{,\theta} w_{,\theta} w_{,x\theta} + \frac{A_{12} + 2A_{33}}{R^2} \bar{w}_{,\theta} w_{,\theta} w_{,xx} + \frac{3A_{23}}{R^3} \bar{w}_{,\theta} w_{,\theta\theta} w_{,x} + 2\frac{A_{12} + 2A_{33}}{R^2} \bar{w}_{,\theta} w_{,x} w_{,x\theta} + \frac{3A_{13}}{R} \bar{w}_{,\theta} w_{,x} w_{,xx}$$

$$+\frac{3A_{22}}{2R^4}\bar{w}_{,\theta\theta}w_{,\theta}^2+\frac{3A_{23}}{R^3}\bar{w}_{,\theta\theta}w_{,\theta}w_{,x}$$
$$+\frac{A_{12}+2A_{33}}{2R^2}\bar{w}_{,\theta\theta}w_{,x}^2+\frac{3A_{23}}{R^3}\bar{w}_{,x}w_{,\theta}w_{,\theta\theta}$$
$$+2\frac{A_{12}+2A_{33}}{R^2}\bar{w}_{,x}w_{,\theta}w_{,x\theta}+\frac{3A_{13}}{R}\bar{w}_{,x}w_{,\theta}w_{,xx}$$
$$+\frac{A_{12}+2A_{33}}{R^2}\bar{w}_{,x}w_{,\theta\theta}w_{,x}+\frac{6A_{13}}{R}\bar{w}_{,x}w_{,x}w_{,x\theta}$$
$$+3A_{11}\bar{w}_{,x}w_{,x}w_{,xx}+\frac{3A_{23}}{R^3}\bar{w}_{,x\theta}w_{,\theta}^2$$
$$+2\frac{A_{12}+2A_{33}}{R^2}\bar{w}_{,x\theta}w_{,\theta}w_{,x}+\frac{3A_{13}}{R}\bar{w}_{,x\theta}w_{,x}^2$$
$$+\frac{A_{12}+2A_{33}}{2R^2}\bar{w}_{,xx}w_{,\theta}^2+\frac{3A_{13}}{R}\bar{w}_{,xx}w_{,\theta}w_{,x}$$
$$+\frac{3A_{11}}{2}\bar{w}_{,xx}w_{,x}^2 \tag{A35}$$

$$L_3^{12}(w,w,w)=\frac{3A_{22}}{2R^4}w_{,\theta}^2w_{,\theta\theta}+\frac{3A_{23}}{R^3}w_{,\theta}^2w_{,x\theta}+\frac{A_{12}+2A_{33}}{2R^2}w_{,\theta}^2w_{,xx}$$
$$+\frac{3A_{23}}{R^3}w_{,\theta}w_{,\theta\theta}w_{,x}+2\frac{A_{12}+2A_{33}}{R^2}w_{,\theta}w_{,x}w_{,x\theta}$$
$$+\frac{3A_{13}}{R}w_{,\theta}w_{,x}w_{,xx}+\frac{A_{12}+2A_{33}}{2R^2}w_{,\theta\theta}w_{,x}^2$$
$$+\frac{3A_{13}}{R}w_{,x}^2w_{,x\theta}+\frac{3A_{11}}{2}w_{,x}^2w_{,xx} \tag{A36}$$

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